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APPLICATION OF VARIATIONAL PRINCIPLES, THE HAMILTON PRINCIPLE IN PARTICULAR, TO THE APPROXIMATE SOLUTION OF HYPERBOLIC DIFFERENTIAL EQUATIONS

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ABSTRACT

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A method of approximate solution is given in general form for combination initial- and boundary-value problems associated with hyperbolic differential equations. The method is based on a dimensional reduction analogous to that of the Hamilton principle in mechanics.

The main difficulty in the hyperbolic, as opposed to the elliptic, case lies in estimating the approximation error and establishing the convergence conditions. The case of a uniform string held at both ends in free oscillation is analyzed as a numerical example, which suffices to demonstrate the merit of the method.

Author

It is well known what a useful role is played in the solution of /689
boundary problems for elliptic equations by their corresponding variational problems. Application of the latter has made it possible, for example, to reduce the approximate solution of a boundary problem to finding the minimum of a function, as in the methods of Ritz and Euler, or to reduce the dimensions of the problem by transforming partial differential equations to a system of ordinary differential equations, as in the method developed by L. V. Kantorovich (see ref. 1, chapter IV, section 3).

Numbers in the margin indicate pagination in the original foreign text.

We will consider hyperbolic equations and problems in which they are involved with initial and boundary conditions. For many such problems, the corresponding variational problems can be asserted. (For problems in mechanics that are expressible in terms of hyperbolic equations, the variational problems can be derived by the least action principle in Hamilton-Ostrogradskiy form.) We will be concerned with the possibility of utilizing variational problems to reduce the dimensions of a problem as part of its approximate solution, much as this is done in the method of Kantorovich.

For definiteness, we will investigate the following problem. Let it be required to find a twice continuously differentiable solution to the following equation in the rectangular region $G\left[0 \le x \le l, 0 \le y \le Y\right]$:

$$(au_x + bu_y)_x + (bu_x + cu_y)_y - F_u(x, y, u) = 0, \qquad ac - b^2 < 0, \tag{1}$$

satisfying the initial conditions

$$\left.\begin{aligned} u(x, 0) &= \varphi_0(x) \\ u_y(x, 0) &= \varphi_1(x) \end{aligned}\right\} \quad 0 \leqslant x \leqslant l \tag{2)-(2'}$$

and boundary conditions

$$u(0, y) = \psi_0(y),\ u(l, y) = \psi_1(y),\ \varphi_0(0) = \psi_0(0),\ \psi_1(0) = \varphi_0(l). \tag{3}$$

Here a, b, c are functions of x and y, continuously differentiable in G, while F_u is the partial derivative with respect to u of a function F(x, y, u) defined in the region $\left[0 \le x \le l, 0 \le y \le Y, -\infty \le u \le \infty\right]$ and sufficiently 690 smooth therein. We assume the existence and uniqueness of a solution and will concern ourselves only with its approximate determination.

To formulate the variational principle corresponding to the combination problem just stated, we start by inspecting the values of the solution $u(x, y)$ on the side $y = Y$ of the rectangle G: $u(x, Y) = v(x)$. We introduce the following second-degree functional with respect to the partial derivatives u_x and u_y:

$$I(u) = \frac{1}{2}\iint_G [au_x^2 + 2bu_xu_y + cu_y^2 + 2F(x, y, u)]\,dxdy. \tag{4}$$

We will also examine the set Φ of functions $u(x, y)$ which are twice continuously differentiable in G and assume the following fixed values at the boundary of G:

$$\begin{gathered} u(x, 0) = \varphi_0(x), \\ u(x, Y) = v(x), \\ u(0, y) = \psi_0(y), \; u(l, y) = \psi_1(y). \end{gathered} \tag{5'-6}$$

The solution $u(x, y)$ of the boundary problem (1)-(3) is a member of the set Φ.

Inasmuch as the differential equation (1) is the Euler equation for the functional (4), this solution, of all the functions in the set Φ, has the property that it will yield a stationary value[1] of the functional $I(u)$.

[1]The converse may not be true. Let there exist in the set Φ a function u for which $I(u)$ has a stationary value. If this problem has a unique solution, then it coincides with the solution of the problem (1)-(3). But the formulation of a variational problem can have several solutions. Among them will be found the solution of the boundary problem. In view of the assumed uniqueness of solution to the problem (1)-(3), corresponding to different solutions of the variational problem will be different functions $\phi_1(x)$ with the condition $u_y(x, 0) = \phi_1(x)$, and the sought-after solution of the problem (1)-(3) will therefore be determined by the form ascribed to the function $\phi_1(x)$ on the right-hand side of equation (2").

This property of the solution is, for the boundary problem, the analog of the familiar Hamilton-Ostogradskiy principle in mechanics. It can be used for the approximate solution of a boundary problem by dimensional reduction.

Let $A_0(x, y)$ be any sufficiently smooth function satisfying the conditions (2)-(3):

$$A_0(x, 0) = \varphi_0(x), \ A_{0y}(x, 0) = \varphi_1(x); \quad A_0(0, y) = \psi_0(y), \ A_0(l, y) = \psi_1(y) \tag{7}$$

and let, in addition, $A_k(x, y)$ be arbitrary smooth linearly independent functions in G, reverting to zero at x = 0 and x = 1:

$$A_k(0, y) = 0, \ A_k(l, y) = 0.$$

We write the linear combination[2]

$$u_n(x, y) = A_0(x, y) + \sum_{k=1}^{n} A_k(x, y) f_k(y), \tag{8}$$

containing in arbitrary functions $f_k(y)$ $(0 \leq y \leq Y)$.

We subject the latter to the initial conditions 691

$$f_k(0) = 0, \ f_k'(0) = 0, \ k = 1, 2, \ldots, n. \tag{9}$$

If we introduce u_n in place of u in equation (4), we obtain a functional that depends on f_k (k = 1, 2,..., n). We choose f_k such that for $I(u_n)$ the system of Euler equations will be satisfied. It is readily seen that these equations can be reduced to the form

$$\int_0^l \{(au_{nx} + bu_{ny})_x + (bu_{nx} + cu_{ny})_y - F_u(x, y, u_n)\} A_i(x, y)\, dx = 0, \quad i = 1, 2, \ldots, n. \tag{10}$$

[2]Specifically, for $\phi_0(x) \equiv \phi_1(x) \equiv 0$, it is convenient to seek the solution in the form $u_n(x, y) = \sum_{k=0}^{n} A_k(x) f_k(y)$.

The equations are linear with respect to the first and second derivatives f_k' and f_k'' of the unknown functions, and the conditions necessary in order for them to be reduced to canonical form can be enunciated without difficulty.

The functions $f_k(x)$ $(k = 1, \ldots, n)$ must be found by solving the system (10) with the initial conditions (9). Once this is done, we can form the right-hand side of (8) and take $u_n(x, y)$ as the approximate solution of the combination problem (1)-(3).

The author knows of no estimate of the difference $u(x, y) - u_n(x, y)$ between the exact and approximate solutions, nor of the conditions for convergence of $u_n(x, y) \longrightarrow u(x, y)$. It is expected, of course, that the investigation of both problems will be complicated by the fact that the quadratic form $au_x^2 + 2bu_x u_y + cu_y^2$ of the derivatives u_x and u_y appearing in the integral (4) alternates sign for hyperbolic equations, in contradistinction to boundary problems associated with elliptic equations. Consequently, it is probably not possible to expect a forthcoming solution to either of the problems indicated.

We performed a numerical experiment, which gave satisfactory results and led us to expect that the above-designated application of the variational principle should prove useful, at least in some cases, for the approximate solution of combination problems associated with hyperbolic equations.

We analyzed the free oscillations of a uniform string secured at both ends $x = 0$ and $x = \pi$ and, at the initial time $t = 0$, having the shape of a parabola symmetrical relative to the straight line $x = \frac{1}{2}\pi$, with a unit deviation at the midpoint of the interval $\left[0, \pi\right]$. The initial velocities of every point on the string were assumed equal to zero.

The functional (4) for this problem has the form

$$I(u) = \frac{1}{2}\int_0^t \int_0^\pi [\rho u_t^2 - T u_x^2]\, dx dt. \tag{B}$$

We sought the solution in the form

$$u_n(x, t) = \frac{4}{\pi^2} x(\pi - x) f_0(t) + \sum_{k=1}^{n} x^{k+1} (\pi - x)^{k+1} f_k(t). \tag{C}$$

Some of the computational results are given below

x/t	0,5	1,5	2,5	3,0
	0,43398	0,89665	0,54873	0,12289
	0,46956	0,87541	0,57024	0,15102
0,5	0,43319	0,90378	0,54161	0,12629
	0,43589	0,89631	0,54822	0,11821
	0,43328	0,89513	0,54623	0,12367
	—0,17395	—0,46979	—0,22321	—0,049261
	—0,22433	—0,41822	—0,27243	—0,072146
2,0	—0,16662	—0,45828	—0,22620	—0,033594
	—0,16706	—0,45954	—0,22516	—0,036944
	—0,17455	—0,47041	—0,22247	—0,046126
	—0,52717	—0,98984	—0,64194	—0,16404
	—0,53030	—0,98865	—0,64400	—0,17055
3,0	—0,52933	—0,98845	—0,64308	—0,17002
	—0,52540	—0,98322	—0,64667	—0,15252
	—0,52669	—0,99032	—0,64111	—0,16798
	—0,086079	—0,25011	—0,11045	—0,024376
	—0,10904	—0,20329	—0,13242	—0,035069
4,5	—0.11767	—0,20662	—0,14084	—0,039571
	—0,081734	—0,23830	—0,10903	—0,022829
	—0,084263	—0,24364	—0,10732	—0,026022

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The first line of each horizontal band gives the exact solutions, the subsequent lines give the approximate solutions for n = 0, 1, 2, 3, respectively. The ratio T/ρ was assumed equal to unity.

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REFERENCES

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